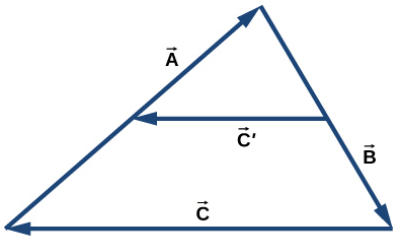
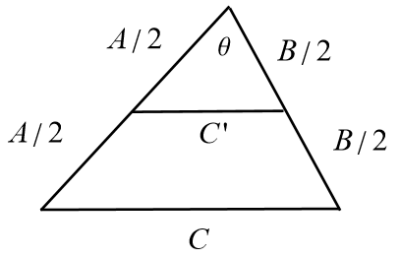
Problem 90

The following figure shows a triangle formed by the three vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$. If vector $\vec{\mathbf{C}}'$ is drawn between the midpoints of vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, show that $\vec{\mathbf{C}}' = \vec{\mathbf{C}}/2$.



Solution

Consider the corresponding vector magnitudes. Let θ be the angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.



Apply the law of cosines to the small triangle and to the big triangle.

$$C'^2 = \left(\frac{A}{2}\right)^2 + \left(\frac{B}{2}\right)^2 - 2\left(\frac{A}{2}\right)\left(\frac{B}{2}\right)\cos\theta$$

$$C^2 = A^2 + B^2 - 2AB\cos\theta$$

Solve the second equation for $\cos\theta$.

$$\cos\theta = \frac{A^2 + B^2 - C^2}{2AB}$$

Plug it into the first equation.

$$C'^2 = \left(\frac{A}{2}\right)^2 + \left(\frac{B}{2}\right)^2 - 2\left(\frac{A}{2}\right)\left(\frac{B}{2}\right)\left(\frac{A^2 + B^2 - C^2}{2AB}\right)$$

$$= \frac{A^2}{4} + \frac{B^2}{4} - \frac{A^2 + B^2 - C^2}{4}$$

$$= \frac{C^2}{4}$$

Take the square root of both sides.

$$C' = \frac{C}{2}$$

$\vec{\mathbf{C}}'$ and $\vec{\mathbf{C}}$ have the same direction. Therefore, $\vec{\mathbf{C}}' = \vec{\mathbf{C}}/2$.